Spoons and hugs: *Whys* reflections

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Once upon a time there was a mathematical investigation that offered a wonderful mirror for reflecting some ideas of Steve Brown and extensions of those ideas. In the mixed image we find one more counterexample of the familiar claim, "You either understand it or you don't."

The spoons problem

The spoons problem and I met a number of years ago. I was wandering in search of good elementary probability problems, and my path led into the Interactive Mathematics Program (IMP) high school curriculum. I turned over the page labeled *Choosing for Chores: Wash or Dry?* (Fendel *et al*, p. 400) and it jumped out at me:

> Scott and Letitia are brother and sister. After dinner, they had to do the dishes, with one washing and the other drying.
>
> They were having trouble deciding who would do which task, so they came up with a method for deciding based on probability.
>
> Letitia grabbed some spoons and put them in a bag. Some had purple handles and others had green handles. Scott had to pick two of them. If the handles were the same color, Scott would wash. If they were different colors, he would dry.
>
> It turned out that there were two purple spoons and three green ones. What was the probability that Scott would wash the dishes? Explain your answer.

The problem and I became friends for several reasons: It tantalized me with the hint of counter-intuition; it involved numbers of combinations; and I hadn't seen it before. As I played with it, I became more deeply involved in the extension: *Under what conditions* (if any) is the probability $\frac{1}{2}$? My intuition said the game would never be fair.

Problem posing reflection

Problem posing is an essential idea of mathematical investigation. This volume includes several accounts of how Steve Brown and Marion Walter have written about *What-if-not?* (WIN) questions (Brown and Walter 1970, 1990; Walter and Brown, 1969). Other question openers can also lead to exciting and illuminating explorations. Some of my favorite starters, besides *Under what conditions?*, are *How many?, What's the most?,* and *In general?*.

As I recently consulted *The Art of Problem Solving*, I was struck by the occurrence of many of these favorite problem openers. They were subsumed under *What-if-not?* investigations, often while brainstorming on one changed attribute. This observation led me to ask several questions, including "Is there a WIN question lurking behind every question?"

A pattern in the spoons

To convince myself that the spoons game would never be fair, I began trying special cases. To my surprise, the game is indeed fair for the fairly simple case of 1 purple and 3 green spoons (or *vice versa*). Under what conditions, then, would it be fair? *What if not* 1 and 3? Further trials led me to see that it was fair for 3 purple and 6 green spoons. Could there be a pattern related to triangular numbers? I tried 1 purple and 6 green spoons without success. But the game was fair for 6 purple and 10 green spoons, and for 10 purple and 15 green spoons. This pattern of consecutive triangular numbers made the problem all the more fascinating to me. *In general*, does the pattern hold?

With some messy algebra, I proved that "If the numbers of purple and green spoons are consecutive triangular numbers, then the game is fair." I found this unexpected relationship so attractive that I convinced a team writing online investigative discrete mathematics materials (Copes *et al*, 2000) to include the problem.

If you're like me, you're asking another very important question opener right now: *Why?*

Why? reflection

We mathematics teachers seem to spend a lot of our time explaining why, whether or not students have asked us to. But we often forget that different people have different criteria for what constitutes an answer to the question *Why?*

My spouse Jane, for example, relates how one of her students in a General Chemistry class asked, "Why is Avagadro's number what it is?" Other students were impatient: "Because it is!" Did the inquiring student want an answer in number mysticism? An appeal to God? Because Jane was less impatient, she finally determined that she could answer his question *Why?* by describing an experiment that determined the number.

This story reminds me of my former student Karen, whose ability to make meaning of new mathematical ideas was remarkable. At one point I asked her why she thought she was finding the ideas of abstract algebra so much more intuitive than her peers did. She said immediately, as if she'd thought about that question before, "I've always been taught that there are reasons why, but most other students don't think there are. Their answers to *Why?* are *Just because*."

Several colleagues to whom I've posed the spoon problem are quite satisfied with seeing the pattern. They say, "Cool," and the question *Why?* is apparently answered for them. For other colleagues, the question is answered as soon as they see a confirming algebraic calculation.

Many years ago I read in Brown (1981a), "The kinds of questions that make sense to me in terms of solidifying understanding are very different from those that make sense to you." He was arguing that teachers can't do *What-if-not?* questioning for their students. Now I understand his claim in a new way.

My colleagues sigh when I ask questions such as "But *why* do you invert and multiply when dividing fractions?" They know I don't want a demonstration of the algorithm; they're sure I won't be satisfied with a few examples; and they strongly suspect that an algebraic derivation won't make me happy. They have learned that my criteria for answers to *Why?* are tied to some meaning I already have about the concept under consideration. For example, for me a satisfactory answer to the question "Why do consecutive triangular numbers of spoons make the game fair?" would relate to the meaning I have constructed of triangular numbers.

A few years later

Recently I was again looking for problems about probability, and I remembered the spoons problem. This time, thanks to some free time on a business trip, I had the freedom to pursue the *Why?* question more deeply.

I reviewed the algebraic derivation. I even proved the converse: The game is fair only if the numbers of spoons are consecutive triangular numbers. Coming up with that proof was sort of fun. When I rewrote the equation ${}_pC_2 + {}_gC_2 = pg$ using the definition of combination numbers and solved the quadratic equation for p in terms of g, I got

$$p = \frac{2g - 1 \pm \sqrt{1 + 8g}}{2}.$$

One condition for p to be an integer is that the term $1 + 8g$ be a perfect square. In fact, since $1 + 8g$ is odd, it's the square of an odd number. Say, $1 + 8g = (2n - 1)^2$. Then

$$g = \frac{(2n-1)^2 - 1}{8} = \frac{4n^2 - 4n + 1 - 1}{8} = \frac{4(n^2 - n)}{8} = \frac{n(n-1)}{2},$$

a triangular number. Substituting for g in the expression for p, I verified that p is either the previous or the next triangular number.

But this proof gave me no insight into *Why?*

Proof refection

Traditionally, mathematical proof has been seen as a method for verifying that a conjecture can fit validly into a given axiom system. But, as Fawcett (1938) and, much later, de Villiars (1999) have pointed out, a proof can serve other purposes as well. It might be used for discovering new relationships, for example, or for communicating results, or for discovery, or as an intellectual challenge. My hope was that it would serve the purpose of illumination—of answering my question *Why?* Elsewhere in this volume Henderson calls such a proof "alive." This proof didn't seem very alive to me, though others might find it quite satisfactory as an explanation.

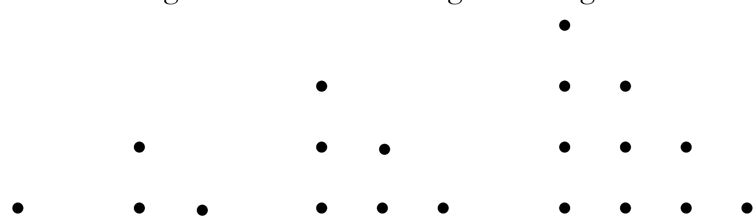
Incidentally, the proof does contain an interesting bonus: If $1 + 8g$ is a perfect square, then g is a triangular number. Thus the proof did serve the purpose of discovery, at least for me.

Is there a WIN question lurking behind all questioning? If so, it's not obvious to me. More clear would be the claim that there's a *Why?* question lurking behind much WIN questioning. After all, the purpose of much WIN questioning is to throw light on the original situation.

Triangular meanings

As I became more aware of what I was seeking—an answer to the question *Why?* that built on the meanings I'd constructed for triangular numbers—I asked myself what meanings I had in fact constructed. Four concepts came to mind:

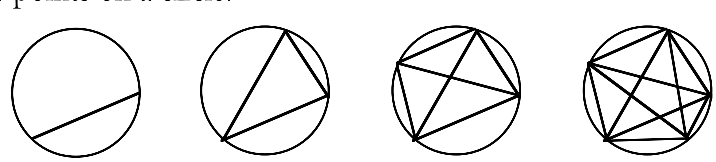
1. the nth triangular number as a triangular arrangement of n dots:



2. the nth triangular number as the sum of the first n positive integers:

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

3. the nth triangular number as the number of chords connecting n points on a circle:



4. the *n*th triangular number as the number of possible different handshakes among *n* people.

I prefer thinking about visual images, so I concentrated on triangular arrays of dots and, to some extent, on chords in a circle. On my business trip, I found myself thinking of dots while in the shower. I dreamt of dots. I bugged my colleagues about dots, to their patient amusement, hoping that my description of the problem would give me, if not them, some ideas. No luck.

One morning as I awoke, I lay in bed thinking about dots and chords. Wandering into no new ideas in that direction, my mind turned to handshakes. Could the problem be rephrased in terms of handshakes? Well, not exactly. But close:

> You have two groups, perhaps men and women. The men all shake hands among themselves. So do the women among themselves. Between the two groups, they do something else. How about hugs? Each woman hugs each man. Under what conditions, if any, does the number of hugs equal the number of handshakes?

It's the same problem, but without the probability. I tried out the revised problem on a colleague over breakfast. We had an enjoyable discussion, though no new inspiration.

Problem solving reflection

Despite the plethora of writing about mathematical problem solving over the past half century—led by Polya (1945) and Schoenfeld (1985), with significant contributions by Lakatos (1976) and Mason *et al* (1985)—I find myself wondering how many students engage in the important aspects of my encounter with what I now call the Hugs and Handshakes problem. For example, what students have the luxury—or permission—to pose new problems? My investigation wouldn't have begun if I hadn't asked *Under what conditions?* and *Why?* As Steve (1981a, p. 35) writes, "We tend to . . . lose sight of the fact that problems are generated by human beings and that such generation makes use of the mind not as a logical machine alone but as an instrument for poetic thought as well!"

For a while, the problem and I became very much part of each other. How often can students live through an obsession like mine? Steve (1981a, p. 33) points out that mathematics teaching seems to separate the object to be learned from the learner, despite Dewey's advocacy of integration.

And what students experience the feelings I had, including:

- frustration
- amazement at insights
- conflict of intuition and reasoning
- texture of discussions with peers
- confusion
- satisfaction of deeper understanding
- exhilaration of apparent progress
- thrill of accidental discoveries
- awe at the beauty of connections

That is, how many students experience the humanness of the enterprise? In the words of Dorothy Buerk elsewhere in this volume, how many students experience *connected mathematical thinking*?

Or, to put the question more positively, under what learning conditions could students get this kind of experience regularly?

A quick, if somewhat glib, answer to this question would be "in a classroom in which a well-prepared, professional, and well-supported teacher has established a culture of curiosity." To elaborate on each point:

A *well-prepared* teacher is one who has personal experience in the humanity of mathematical investigations; who has ideas about how to inject that humanity into students' lives; and, as Dan Chazen points out elsewhere in this volume, who has habit and ability of reflection.

A *professional* teacher is one who is not following a script but rather is making decisions on the fly about the best ways to respond to whatever situation arises. Brown, Rising, and Myerson (1977) point out that the professional teacher is the control center of the classroom—a classroom that is teacher-centered so that it can be student-centered. And, the decisions to be made are legion; see Copes (2000) for an example of how a professional teacher might think in the midst of active learning.

Teachers are *well-supported*. Their supervisors and the parents of their students trust their professionalism. Their curriculum, treated as a guide, at least does not hinder spontaneous investigations. Their goals for students are accepted as being more than passing standardized tests and are not viewed as a distraction from high test scores.

Such teachers can establish a *culture of curiosity* in their classroom. Students want to know. They want to be engaged in, even obsessed with, investigations. At the surface, the class may be organized in a problem-based manner (Copes and Shager, 2003, p. 197ff). But more importantly, in a different dimension, students are valued as human beings. As Steve (1981a, p. 36) wrote,

> Though open math environments have tended to encourage students to engage in mathematical activities in a more exploratory and less repressive way than has been the case in many traditional settings, even such experience has tended not to honor dialogue as a genuine open interchange in which the teacher as well as student can hope to increase his or her awareness of self along such dimensions.

Through such interchanges, students learn to see things from various perspectives—not only an important problem-solving technique but also what Lukinsky elsewhere in this book calls "a basic component in any system of moral reasoning."

My own early research focused on bringing students to see that a variety of approaches to a mathematics problem can be legitimate. In rereading Brown (1981b), I'm reminded that such an approach might be counterproductive. In referring to Perry's scheme of intellectual development, Steve writes (p. 17) that we shouldn't limit attention to what the student knows and what we want the student to know, but on students' conception of what knowledge is. "For students who are at a low level of conception of knowledge, how confusing it must be to be told not only that there are several different ways of viewing a problem but furthermore that all of them are correct."

Shaking hugs

In retrospect, it seems that every free minute of the business trip, and some of the un-free minutes, found me enmeshed in the hugs problem, trying to answer the question *Why?*

I went to sleep thinking about points on circle.

As I walked, I imagined triangles of dots. Lots of triangles.

While eating alone, I messed around with algebra. The algebra led me to the fact that the sum of two consecutive triangular numbers is a perfect square. When I drew the triangles of dots, I thought, "Duh." (In fact, as colleagues have pointed out since, the algebra led them to the sum of the numbers of men and women is the square of their difference.)

I was consumed. The problem and I became one.

One person to whom I posed the problem said immediately, "There's not enough information. How many times did they hug or shake hands with each other?"

Curious reflection

Oh, yes. The statement of the problem, as I'd made it, doesn't ask *Under what conditions?* or *Why?* It asks *Is it possible?* Teachers and textbook authors tend to be very touchy about questions like that. "The student can answer 'yes' and be right," they explain patiently to me. "You have to add (at least) the instruction 'Explain' or 'Justify your reasoning.'" I require a lot of patient explanation, because I keep wanting to leave something to the interaction between a student and a professional teacher. I want the student to ask how many times each person shakes hands with or hugs another. I want the classroom climate to be such that of course everyone will discuss each other's explanations to construct meaning. I want the students' curiosity, perhaps prompted by the teacher, to bring up the questions *Under what conditions?* and *Why?* I don't want the students to be "working to rule" and answering only the questions asked.

Elsewhere Kay Shager and I (2003, p. 202ff) proposed a teaching model consistent with this kind of classroom. Imagine you are the manager of a team of investigators in some organization. People (perhaps members of your own group) bring you problems to be solved. As a manager, you are an experienced problem solver, but of course you don't know how to solve these problems in advance. You assign them to individuals or groups within your team. You can point out common problem-solving techniques; you can recognize and reward good thinking; you can help groups work together well, capitalizing on each other's strengths and ideas; and you can help your team members deal with their discouragement and confusion as well as celebrate their awe and exhilaration. What you can't do is tell them how to solve the problems. You want to establish the understanding that, if you knew the only right way to solve the problems, there'd be no point to students' working on them.

One might criticize this model as being unrealistic for a classroom. After all, managers can hire and fire in ways teachers can't. And teachers are supposed to know how to solve all the problems. I can counter the hiring/firing issue only by acknowledging it as a limitation of the model—a limitation that excellent teachers get around. As for teachers' knowing how to solve the problems: Well, we don't. We might know several solutions that we or others have come up with; but we don't know how this student or group of students will solve the problem. The story of my own investigation shows that solving is much more than the solution itself.

Progressive hugs

Toward the end of my business trip, I was napping in the passenger seat of a car. As I was waking up, my mind turned back to hugs. I imagined a room with some men and women. Another person walks in. What happens? How many more hugs and handshakes must there be?

There's a handshake with each person of the same sex. And there's a hug with each person of the opposite sex.

I needed to be more specific. What if there was one person of the same sex and three of the other sex, so that the numbers of handshakes and hugs so far had been the same? A new person arrives. One more handshake and three more hugs. The number of hugs is ahead by two.

Now I imagined another person arrived, of that same sex. Still three more hugs, but now two more handshakes. The number of new hugs is still more than the number of new handshakes, but not by so much. And with a third arrival of the same sex, the number of new handshakes equals the number of new hugs. Now the sizes of the subsets are equal. If another person arrives, the number of new hugs and handshakes will be the same. If another person of that same sex arrives, then the number of handshakes will actually exceed the number of hugs by one. Another person of the same sex will have even more handshakes than hugs.

Where does the total excess catch up with the total deficit? When the size of the growing group exceeds the size of the fixed group by one more than the difference between the original two sizes. In the case that the original sizes were 1 and 3, we had deficits for 2 new people, then a person with the same number of handshakes and hugs, and then excesses for 2 more people, so we added 2 + 1 = 3 new people to achieve a new balance between handshakes and hugs.

This felt like progress. It reminded me of the dynamic strategy of problem solving: Imagine motion, change. People enter and shake hands and hug.

But I was still not satisfied. The question remained, Why triangular numbers? Each number of handshakes is triangular. But why are the numbers of men and women triangular? Those numbers are not even the number of dots on the longest row of the handshake triangles. I was overwhelmed by what a colleague later termed the "four-dimensionality" of the problem. In fact, I seriously tried to think about multi-dimensional prisms, without much success.

Story reflection

Steve (1981b, p. 11) writes poignantly:

> One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. . . That event can . . . act as a peephole through

> which we get a better glimpse of a world that surrounds us but that we may never have seen in quite that way before. Tales, fables, good literature and poetry do this for us. They force us to see what is around and inside all of us by inviting us to examine quite closely a situation that is at first blush both extraordinary and removed.

This paper is based on the assumption that, similarly, we can learn a great deal of the world of mathematics from one story of one rich investigation. Elsewhere (Copes, 1997) I have argued that we need to share our stories with each other and with our students. Even the unproductive and frustrating points in the story (such as my thoughts in the car) can be illuminating, if for no other reason than to reassure each other that none of us is perfect.

My own story also includes learning from Steve Brown. As I reread pieces I haven't looked at in years, I came to realize that long ago he planted the seeds for ideas I've just recently had. His and Marion's *What-if-not?* analyses gave rise to my own problem openers. He pointed out the impossibility of building meaning for students. He trumpeted the need for learners to fuse with the mathematical ideas they are constructing. He reminded me that the possibility of more than one legitimate approach to a problem can induce trauma in students whose world is, in Perry's (1970) language, "Dualistic." And Brown, with Rising and Meyerson (1977, p. 195), reminds me of the importance of conversation:

> To the well-known line
>
> > I do and I understand
>
> we add the words "only superficially" and the additional line
>
> > I discuss and I make my own.

Perhaps most importantly, however, Steve can't leave alone the creativity of human beings. Is answering *Why?* a creative act? Is it part of our human-ness to *create* meaning?

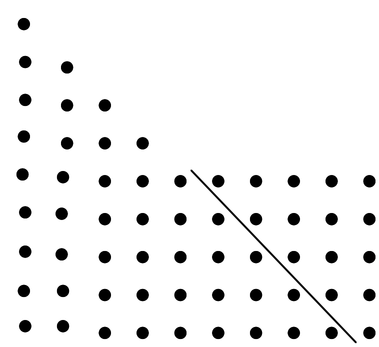
How much my story has been enhanced by what I learned from Steve—even without my being very aware of it!

Erik's idea

An important character in my story is Erik. He and I have been friends for over a decade, since he was 12. When he was in high school, we would get together weekly to discuss mathematics problems with which my colleagues and I were challenging our college students. Home recently from graduate school in music, he called to set up lunch with me. Over soup and sandwiches, I offered him the problem of handshakes and hugs. Without any prompting, he came up with several familiar ideas without working out details.

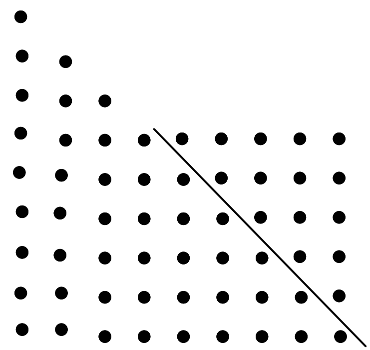
Like me, he expressed dissatisfaction with the algebra. He wanted, he said, a picture, a unifying image that explained in a glance. As is often the case when Erik and I convene, we spent much of our time thinking in companionable silence.

The next day Erik sent email with an idea. Illustrating with the case of groups of 10 and 6, he suggested placing the triangles representing the corresponding numbers of handshakes (T9 and T5) next to each other, one upside down, like this.



Then, he pointed out, you almost have a rectangle representing the hugs. It's missing a row, but that row is exactly in the triangle sticking out above the rectangle.

I could prove algebraically that his observation would always hold if the numbers were triangular to begin with. In fact, if the smaller triangle were shifted up one row and to the left one column, the excess would exactly fill in the missing column.



And now I could reason out the converse. This scheme can be applied to any numbers of men and women; but, if the number of hugs exactly equals the number of handshakes, then the leftover triangle must fill either the missing row or the missing column, so the number of men and the number of women must be triangular.

Thus Erik supplied me with a picture. Indeed, he gave an explanation in terms of one of my meanings of triangular numbers—as a triangle of dots. It's an elegant solution, and several of my friends have found it quite compelling.

Much as I like pictures, however, I found that I still wasn't satisfied. Apparently I wanted something more than an explanation in terms of one of my understandings of triangular numbers. What did I want? Eventually I decided that I wanted an explanation involving actual handshakes or hugs, not just dots.

Opensure reflection

So from this investigation I have learned that my criteria keep changing. My experience is consistent with Steve's belief that "it is a serious error to conceptualize of mathematics as anything other than a human enterprise which among other things helps to clarify who we are and what we value. (1981a, p. 27)"

In personal conversations, if not in writing, Steve has decried teachers' and students' need to bring closure to a problem or class session. He has pointed out that closure signals that there's no more to be thought about. He coined the term "opensure" as a preferable goal.

In the spirit of opensure, I'm going to end this description of my spoon and hugs journey, although my own travels didn't end here. There's more thinking we can all do—more meanings we can make—of the mathematics involved in this problem. And there's more understanding to be accomplished of the reflections Steve Brown has invited us to share.

References

Brown, Stephen I. (1981a). Ye Shall Be Known by Your Generations. *For The Learning of Mathematic* s 3 (March): 27-36.

Brown, Stephen I. (1981b, March). Sharon's Kye. *Mathematics Teaching* 94 (Winter): 11-17.

Brown, Stephen I. and Marion I. Walter (1970). What If Not? An Elaboration and Second Illustration. *Mathematics Teaching* 51 (Spring): 9-17.

Brown, Stephen. I and Walter, Marion I. (1990). *The Art of Problem Posing*. Second edition. Hillsdale, NJ: Lawrence Erlbaum.

Brown, Stephen I., Gerald R. Rising, and Lawrence N. Meyerson (1977). The Teacher Centered Classroom. In *Organizing for Classroom Instruction*; *NCTM Yearbook*: 186-197. Reston, VA: National Council of Teachers of Mathematics.

Copes, Larry (1997). Stories of investigations. *Primus* VII, 1 (March): 43-61.

Copes, Larry (2000). Messy monk mathematics: An NCTM-standards-inspired class session. *Mathematics Teacher* 93, 4 (April): 292-298.

Copes, Larry, Joan Lewis, Benjamin Cooper, and Nancy Casey (2000). *Discrete Mathematics Investigations for Teachers*. SciMath Minnesota. home.comcast.net/~lcopes/SciMathMN/

Copes, Larry and N. Kay Shager (2003). Phasing Problem-Based Teaching Into a Traditional Educational Environment. In Shoen, Harold L. and Randall I. Charles (eds.), *Teaching Mathematics through Problem Solving: Grades 6-12*, pp. 195-205. Reston, VA: National Council of Teachers of Mathematics.

de Villiers, M. (1999). *Rethinking Proof with Sketchpad*. Emeryville, California: Key Curriculum Press.

Fawcett, H. (1938). *The nature of proof*. New York: Teachers College.

Fendel, Dan, Diane Resek, Lynne Alper, and Sherry Fraser (1999). *Interactive Mathematics Program Year 3*. Emeryville, CA: Key Curriculum Press.

Lakatos, Imre (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. New York: Cambridge University Press.

Mason, John with Leone Burton and Kaye Stacey (1985). *Thinking Mathematically*. New York: Addison-Wesley.

Perry, William G., Jr. (1970). *Intellectual and Ethical Development in the College Years: A Scheme*. New York: Holt, Reinhart, and Winston.

Polya, G. *How to Solve It: A new aspect of mathematical method*. Princeton University Press, 1945.

Schoenfeld, Alan H. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Walter, Marion I.. and Stephen I. Brown (1969). What If Not?. *Mathematics Teaching* 46 (Spring): 38-45 .

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