

Messy monk mathematics: An NCTM- Standards-inspired class

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Host teacher is introducing me. Glad they all have name tags. The class looks lethargic. Hope I can wake them up. I'm on. Stand and smile.

I'm really delighted to be here, although you might prefer to be taking an afternoon nap.

Good. Some laughter. Not much, but a little is better than none. Pause. Set the mood. Stare into space. Try to get them wondering what I'm up to.

Once upon a time there was a monk. He lived at the bottom of a not-so-high mountain in . . .

Short pause. Mary is looking a little startled. Bruce appears to be skeptical. Shrug my shoulders.

. . . Tibet? He has this strange habit. So to speak.

Good. Another titter. Who was that? Jane?

Precisely at sunrise on the last day of every month, he leaves his hut at the bottom of the mountain. He walks up a path to the top of the mountain, timing it so that he arrives at the top precisely at sunset. Got that?

A few nods. Bruce is still skeptical. Sara there on the front row is nodding hard. Wonder if she's heard the problem before. Hope not.

I need you to nod or shake your head, if only to stay awake. Got it?

Yeah. Everyone's nodding except Jon.

Jon? Maybe I was looking elsewhere. Are you OK? Good. The monk meditates all night, or whatever monks do. The next morning, the first day of the new month, he leaves the top of the mountain precisely at sunrise. He walks down exactly the same path to the bottom of the mountain, arriving back at his hut precisely at sunset. OK? Nod or shake.

They're all nodding, except Sara. She's sitting back, looking smug. What do I do if she can already solve the problem? Even though she's not nodding, I don't dare ask her if she understands. Pause for the dramatic moment.

The question is, "Is there necessarily a point on the path at which the monk arrives at the same time of day on both days, both on his trip up the last day of one month and on his trip down the first day of the second month?" We'll concentrate only on one round trip, forgetting other months.

Don't know if I should make that last stipulation, but when I don't it leads to all kinds of confusion. Confusion is not necessarily bad, but in this case it seems to distract from the investigation. Considering more than one round-trip can always be used as an extension. Oh, no! Sara's hand is up. If I call on her and she announces her result now, it could cause real difficulties. When people hear an answer, especially from someone who's probably always correct, they stop thinking about the problem, even if they don't understand the ideas very well. I'll try to ignore her for a minute and get some others involved.

Now, at first, I just want your gut reaction, your intuition. Mary?

I don't know.

Good. If you knew, I'd be intimidated by your brilliance. At this point I just want intuition. Intuition is important in mathematics, though it has to be checked against logic eventually. What's your intuition?

No.

Most people who understand the problem would agree. Mary must not be assuming a constant pace. Hope I find someone who does right away, or they'll reconsider their response and that possibility won't be raised. But first, I need to give Mary some reinforcement. And repeat the statement to emphasize the quantifiers.

You're saying that no, there's not a point on the path at which the monk arrives at the same time on each leg of his trip?

Sara's hand is still up. She's pretty eager. I shouldn't really ignore her much longer.

Yes. I mean, yes, that's what I'm saying.

Thanks, Mary.

Maybe one more before Sara. Jennifer is looking as if she might tune out.

Jennifer, what's your gut feeling? No defense is needed at this point.

No.

Good. Carl is shaking his head. He may bring up the midpoint. But I've got to let Sara go next. Her hand is really waving.

Thank you, Jennifer. Sara?

Yes, there has to be a point. Imagine two monks, one going up and the other coming down. They have to cross, and it will be at the same place on the path at the same time of day.

Very clearly put. What now? The teacher's eyebrows are raised. Thinks the investigation is over. Maybe it is. But if I just stop when someone says an answer I'm happy with, I'm violating the spirit of inquiry. Let's find out what the others think of her idea.

Who agrees with Sara?

Oh. They're looking puzzled. No hands up. Aha! Filters! Maybe they're filtering out what they're not ready to understand.

Does anyone understand what Sara just said?

Still no hands. A lot of blank looks. Maybe this won't be such a short session after all! But Sara could feel she needs to explain more, and that could stop others' thinking. I don't want her to say more yet, but she needs some encouragement.

Thank you, Sara. Hold that thought. You've probably planted a seed. Carl, what's your intuition?

The answer is yes. The point will be the midpoint.

I want to get more people involved before we begin justifying. But that seed is planted, too.

Thank you, Carl. Lesley?

Oh. I was going to say no, but now I'm not sure.

Is it Carl's magnetic personality or something he said that cast doubt in your mind?

I hadn't thought about the midpoint.

But your gut feeling was no?

Yes.

Thanks, Lesley. Tom?

I think the midpoint is the point.

OK. Jon?

My gut feeling is no.

Good. Cathi?

Did you say that the monk walks at a constant pace?

Should I answer that question or repeat the problem? I think I should repeat it. They need to know it well, especially the quantifiers in the question. And that gives us another chance to focus on just two days.

Did I? Let me repeat what I said. The monk leaves his hut at the bottom of the mountain precisely at sunrise the last day of the month. He walks up the path, timing his journey to arrive at the top precisely at sunset. The next day, he leaves the top precisely at sunrise, and walks down the path, arriving back at the hut precisely at sunset. Is there necessarily a point on the path at which he arrives at the same time of day on each of those two days? What's your intuition now, Cathi?

No.

She settles back comfortably. Maybe she has some justification in mind.

OK. Thanks. You back there, the woman with the red t-shirt. Sorry, I can't read your name tag. Sue? Thanks. What's your gut feeling about the monk, Sue?

I don't like him.

Good. Everyone's laughing. She is funny. Look chagrined.

Thanks for sharing that, Sue. Would you like to talk about it?

They're more relaxed now.

Not really.

Does she replace thinking with humor? Another chance to restate the problem.

Feel free any time. To rephrase my question, what's your intuition about whether or not there's a point on the path at which the monk arrives at the same time of day on each of the two days?

There's no such point. Sunrise and sunset change.

Aha! She thinks! Cathi's nodding vigorously. I'd better acknowledge that she had the same idea.

Thanks, Sue. Was that your idea too, Cathi?

Yeah. The days aren't the same length.

And. . .

That was José Good. Wish he hadn't stopped to wait for my permission, but I can't expect otherwise this early in our first session together.

José?

Maybe if I sit one the edge of the table they'll feel that they should converse with each other.

Also, the times of sunrise and sunset are different at the top of the mountain and at the bottom.

Good thinking. That point doesn't always come up. On the other hand, everyone seems stunned. Oh, there goes Sara's hand again. They're probably mostly convinced that there's no point, so it wouldn't hurt to have her come in again. We're moving into justification, even though not everyone has expressed a gut feeling. Jon's hand is up, too, but I want to get to everyone before too much repetition. Otherwise a small bunch will take over.

Ooh! Good thinking, José. Despite the fact that I was looking for gut feelings, several people have given some justification for their intuition, looking for a proof.

There. I slid in the idea that proof is justification. Now maybe I can use the word "argument" synonymously. Some folks think of argument as conflict, to be avoided. Sara's hand is down.

One argument that there is a point is that it's the midpoint. Who agrees?

Few hands. Time for quantifiers. Who hasn't said much?

Do you disagree, uh, Bruce? Why?

I don't know.

Jane?

I'm not sure, but the question was, "Is there *necessarily* a point?" If he walks at a constant pace, there *might* be a point—the midpoint. But if he doesn't, I don't think there's *necessarily* a point.

Was that clear? Some nodding. Sara's hand is back up. Maybe I should rephrase it, but I don't want it to sound like a pronouncement from me, which could indicate that we're finished.

Are you saying that to prove there's necessarily a point, we have to consider all cases?

Well, yeah.

So that would make the constant pace situation a special case?

Yeah.

Lots of nods. Sara's hand is down.

Often in working on a problem it helps to consider special cases. So what about this special case in which the monk's pace is constant? Do we have a solution then? What do you think, Amin?

Sure.

Do you agree, Tina?

Uh-huh.

Not sure either is really thinking. She at least is probably taking the easy way out. Who was it who first raised the sunrise and sunset times?

Sue?

Yes.

She's supposed to object!

Yes what?

Yes, if the monk's pace is constant then the midpoint is the point.

Who else?

José?

I don't think so. Sunrise and sunset will still make a difference, even if the pace is constant.

Thank you, José.

But don't sunrise and sunset change symmetrically?

That was Sue. That's why she agreed with the claim about the special case. I'm glad they're not relying on me as a mediator. But I wonder if I can push her to show us a graph or something.

What do you mean, Sue?

Yes. She looks as if she wants to draw something. I bet she's enough of a ham not to resist leading the way.

If you'd like to use the board, feel free.

Of course she's hesitating. Get a marker and hand it to her.

The board is our collective scratch paper.

Yes, she's eager. Oh, she's going to write some formulas.

Distance is rate times time, right? So the first day let's say you go a distance of 6 miles. And you take 12 hours. So your speed is 1/2 miles per hour.

That's average speed.

Thank you, Tina, for waking up.

Oh, yeah. But speed is constant in the special case. So at the midpoint you've gone 3 miles in 6 hours. And the second day you go faster, right, because you're going down hill. Like, 6 miles in 10 hours, or 6/10 miles per hour. And you're halfway down, 3 miles, in 5 hours. Like, at the midpoint.

But that assumes you're going at a constant speed.

Well, yeah, Terry. But it also misses another point. . . .

You can't take less time the second day. You still have to travel from sunrise to sunset.

Thanks, Troy. They seem to be conversing well without me. I'll just sit back on the edge of the desk.

I don't think sunrise and sunset are symmetric. I heard that sometimes sunset comes earlier even though the days are getting longer, so sunrise is much earlier.

I've heard that too, Melissa. Wonder if anyone will agree.

It doesn't matter. Look, if you have two monks, then they'll meet no matter when they leave the ends of the path.

Yeah, Sara now feels free to participate without being called on. Time for a little more control. Stand up. I'm glad she's on the front row so I can reinforce her without others' noticing so much. They're beginning to talk among themselves anyway.

Thanks, Sara. I like your thinking.

There's definitely not a point.

Keep talking to Sara.

But apparently you haven't convinced everyone yet. Keep trying, and listen to see if someone else has a better idea.

Now to take advantage of that "definitely not" that came out of the class somewhere. Need to get to the objective of the exercise, concerning proofs and justification.

There's definitely not a point? That's pretty strong. I asked if there was *necessarily* a point. I have heard arguments that there *is* necessarily a point if the monk's pace is constant; in fact, that point is the midpoint. I've heard the claim that he can walk at a varying pace and avoid having

such a point. I haven't heard an example of such a walk yet. And now I heard someone claim that there was definitely no point. To support that argument you'd have to show what was wrong with the case of the constant pace and midpoint.

I meant that there wasn't necessarily a point.

Thanks, Jane. Does anyone believe that there is definitely no point at which the monk arrives at the same time of day on each of the two days?

Silence. I think they'd all feel comfortable saying so if they believed it, but perhaps I'm wrong.

Well, let's take a vote between definitely yes and not necessarily. All who think there's definitely such a special point, raise your hands.

Sara now has company in Maurice. Wonder what his idea is. He looked pretty eager in voting; I imagine he isn't assuming the constant pace.

All who think there's not necessarily a point?

Not everyone else. Guess we need a third option after all.

All who don't think?

Again some laughter. Controversy engages them. But need to get them more open to Sara's opinion.

Great! I love controversy! But since so many of you agree, you must be right. Right? OK. If it's so obvious, maybe you can prove it. What's involved in a proof that there's not necessarily a point? Terry?

I have no idea.

OK. Chris?

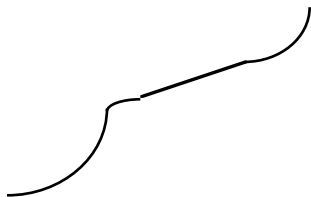
Just show how to walk up and down the mountain.

How could you show it?

Maybe make a chart. Or draw a picture.

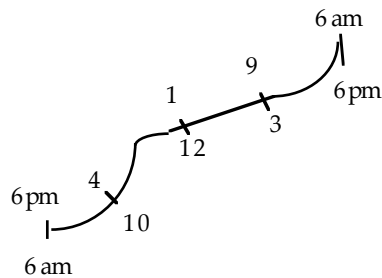
A picture? How? Would you come up and draw one?

Good. No hesitation. Chris draws



Maybe you can label under the line the times for the monk going up, and above the line the times for the monk going down.

Chris draws, hesitatingly,



Great idea, Chris. I want everyone who thinks there's not necessarily a special point to draw such a picture. Please get together with your neighbor and come up with a picture that supports your claim. Sara and Maurice, can you conspire to produce an argument that will support your position? Lori, why don't you join Troy and Amin, since we have an odd number.

Lots of noise. Good. Wander around. Everyone's involved. Some groups challenging their own pictures. Josh and Derek arguing. Troy and Amin seem to be ignoring Lori. Terry and Chris look very puzzled. Andi and Mary are working individually. Hope they get around to looking at each other's pictures. Kerry and Toni look satisfied. Have them put their work on board. Others will challenge it. Sara and Maurice are discussing a graph as well as two-monk model. The teacher is missing some interesting ideas by remaining seated. We have only 20 minutes left. Difficulty in getting them back together means they're involved.

We have a picture on the board. Is everyone happy with it?

Good silence. Nobody's taking a nap. Feel the wheels turning. Lots of hands. They're not speaking out spontaneously any more. Amin hasn't said much, but Lori needs some attention.

Lori?

What about between 12 and 1? I think if you break that down, you'll find a point where he passes at the same time of day each day.

Good. Using the same phrase. Say so.

I'm flattered that you used my words. Thank you, Lori. Amin?

I agree.

Thanks, Amin. Kerry and Toni, can you help Lori see why there's no point?

Good that they both are walking up to the board. But they don't understand.

What?

Can you give details about what happens between 12 and 1, so others will be convinced by your picture?

They could draw enlarged picture, but they're trying to squeeze in the numbers on the original picture. Oops, they switched the order. Time is going backward on the top. Should I correct? It's better if one of the students catches it.

Oops, we switched the numbers.

Thank you, Toni. Saved being corrected by someone else. Can they see it now? Apparently. They're looking puzzled, especially Kerry.

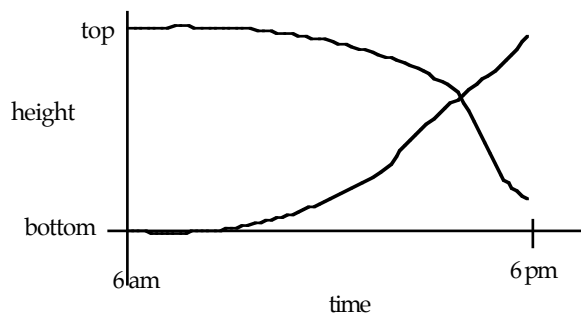
Did anyone else get a picture you're happy with?

No hands. Now Maurice is tentatively raising his. Should he and Sara go? Maybe with the graph.

You folks had a graph, didn't you Maurice? Would you please draw it for us?

OK.

He's shy, but he's doing it. Sara doesn't look offended. Maybe she's sure she'll get her chance.

**You can't get from here to there without crossing the other line.**

Silence. Let it sit for a minute.

No. That's the same point on the path, but it's not the same time.

Thanks, Carl. If nobody had said that, I'd have to play devil's advocate, since I'm sure some people are wondering.

And the mountain just goes one way.

Oh, yeah. Lesley. Also confused about the axes. It's tricky. Can Maurice explain? He's taking his time to consider.

You have to think of what's on each axis. The x-axis represents time of day, not just time. And the y-axis represent height on the mountain. Neither graph is a picture of the path. They're graphs of height compared with time of day.

Good explanation, though there's no x or y in the picture. Who gets it? Carl is nodding. Derek looks very confused. Can't tell about Lesley.

Derek?

I don't get it. What do you mean by height on the mountain?

It flowed right out. Not wondering why I asked. We're communicating. Should I explain? Or suggest to Maurice. . . .

Maurice, do you mean that the vertical axis represents distance along path from the bottom?

Sure.

Should I have said that? Is it too leading? Telling is rarely teaching. It doesn't do much good to answer questions that the students haven't asked. On the other hand, filters will probably work. Oh, there's Josh. He was pretty adamant before that there was no point.

I don't know how to draw a picture, but I like Sara's idea about two monks.

Really? He understood it? Is the pendulum swinging toward thinking there's a special point?

Can I do a demonstration?

Sure.

Sara doesn't look offended. It was kind of Josh to give her credit.

Bruce, can you get at that end of the room, and Mary, at this end? Now, Bruce is the monk going up the mountain, and Mary is the monk going down the mountain the next day. So start walking toward each other. See? When they meet, it's not the same time, because they're on two different days.

Oh. He's pretty confused. But what a wonderful intermediate step! Will there be controversy? Andi's eager.

Of course they meet at the same time. When else would they meet?

Kind of chaos now. Jane, Lesley, and José seem most eager to point out Josh's error. Wonder if they realize that they're convincing themselves of the weakness of their earlier position. Don't want Josh to be turned off. He's looking a little dazed. Not much time left, though. Perhaps they're ready to hear Sara.

Thank you, Josh. Maybe Sara can answer the challenges. Sara?

She looks confused. Of course, she's agreeing with the challenges. Keep her thinking, though. I suspect she can handle the confusion.

As Josh said, imagine that you have two monks, one at the top and one at the bottom. Like Bruce and Mary. Bruce follows the original monk's every movement on the first day, going up the mountain. Mary follows the original monk's movement on the second day, coming down the same path. Even with changes in sunrise and sunset, or even stopping for lunch, the two monks have to meet. They'll meet at a point where the original monk arrived at the same time of day on both days.

Lots of nods this time. But it's not unanimous. Heather still isn't convinced. Filters haven't stopped working. Given time, I'd survey individuals to see what they think. and question Heather more. But I'll have to skip a step, because it's not my own class.

Nice, Sara. Who thinks Sara's two-monk model is equivalent to the original problem? Melissa?

No.

How many think that one monk on two days is equivalent to two monks on one day?

We're having to go too fast. Only about half. Sara is looking astounded. Perhaps she's thinking that her peers are pretty dumb. But they're thinking well; they just aren't ready to see

her approach. Anyway, some of those who don't believe the two-monk model may have been convinced by the graph.

Either because of the two-monk model or the graph, who now thinks that there's necessarily a point on the path at which the monk arrives at the same time of day each of the two days?

Aha! Almost unanimous. Who doesn't agree? It doesn't matter now. I have to stop. Only about a minute left.

Great. Still some disagreement. Keep messing with it. I'll mention briefly that you may want to relate this to a theorem called the Intermediate Value Theorem.

Glad I took a few seconds to write that on the board. Some people actually are copying it down.

But there's one more step. We've been deepening understanding on the forefront of our own mathematical knowledge. It's been pretty messy as our intuition has clashed with our logic. The same kind of mess happens when mathematicians do research, building understanding on the frontiers of knowledge of the mathematical community. So you have experienced some of the frustration and exhilaration and perhaps even beauty of doing mathematical research.

When doing research, unlike when studying mathematics traditionally, we don't stop when we solve a problem. Every solution leads to more questions, making more research in the future. What questions can we ask about the monk problem? One question might be, "How do we prove that those two lines in the graph have to cross?"

Will they come up with questions? It's difficult to ask open ended questions you can't answer. Wish we had more time to do it. Maybe just one tool. . . . What can they do if I take an extra minute or two? At least nobody seems eager to leave.

Often good research questions begin with "What if. . .?" Asking *what if* can impress the heck out of all your teachers.

Good. More laughter. But will they have any ideas?

What if the monk walked backward?

The clown with a brain can still make people laugh.

Good. Not exactly a *deep* question. . .

Be sure to grin broadly. Lots of laughter now.

. . . but a researcher might ask lots of questions before getting one that leads somewhere. Troy?

How do sunrise and sunset really differ?

Pretty. Kerry?

Are the endpoints actually points on the path? If not, then do the monks really need to meet on the path?

Oooh. Nice question. Tom?

What if we consider more than one month, more than two days?

Lots of heads shaking. They're still engaged. Should I wrap it up? Nah. Someone once said that closure marks the end of thinking. If you feel you have to summarize, do it the next day.

Excellent. Now you have lots to think about for awhile. Let me know what your thoughts are.

My students never applaud like this at the end of a class session! But the clapping is somewhat distracted; some of the students are still more interested in the problem than in me. Guess they woke up.

This piece is dedicated to Stephen I. Brown, an inspiration for inquiry-based teaching and learning, on the occasion of his retirement.